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Stringy Cosmic Strings and Compactifications of F-theory

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ABSTRACT

We construct stringy cosmic string solutions corresponding to compactifications of F-theory on several elliptic Calabi-Yau manifolds by solving the equations of motion of low energy effective action of ten dimensional type IIB superstring theory. Existence of such solutions supports the compactifications of F-theory.

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1 Introduction

Recently, ‘string dualities’ among various superstring theories including M- and F-theory have been greatly investigated and several phenomena related to non-perturbative behavior of string theory have been found [1]. F-theory [2] is considered as a twelve dimensional theory underlying the $SL(2, \mathbf{Z})$ symmetry of ten dimensional type IIB theory. This theory may be considered as a key ingredient of the ‘duality web,’ though the corresponding microscopic theory has not been constructed.

Compactification of F-theory on an n -dimensional Calabi-Yau manifold (Calabi-Yau n -fold) X makes sense if X admits an elliptic fibration over some $(n-1)$ -dimensional base B . It is defined as compactification of type IIB theory on B with the field $\tau \equiv \tilde{\phi} + ie^{-\phi}$ varies on B , i.e., $\tau = \tau(z^i)$ ($\{z^i\} \in B$). Here ϕ is the dilaton, $\tilde{\phi}$ is the scalar in the R-R sector and τ is identified with the complex structure modulus of the fiber torus.

For example, consider a compactification of F-theory on $K3$ surface which admits elliptic fibration over the base \mathbf{CP}^1 . In terms of type IIB theory, this model is interpreted as a compactification of ten dimensional type IIB theory compactified on \mathbf{CP}^1 with 24 D 7-branes [3] located on it. We have so called stringy cosmic string solutions of type IIB theory such that ten dimensional spacetime is $\mathbf{CP}^1 \times \mathcal{M}^8$ and generically on 24 points in \mathbf{CP}^1 the field τ diverges. On the other hand, from the F-theory point of view, we understand this property of τ from the geometrical structure of elliptic $K3$ surface. Moreover, the structure of moduli space of the theory such as symmetry enhancement or massless spectrum can be deduced only from the geometric data. The resulting eight dimensional theory is conjectured to be dual to heterotic string theory compactified on torus. One of the evidences of this duality is that [4] there is a relation between F-theory on $K3$ near the orbifold limit of $K3$ and T-dual of type I theory on torus (type I’ orientifold theory).

Compactifications of F-theory to lower dimensions have also been investigated so far [5, 6, 7]. In particular, six dimensional theories obtained by compactifications on several elliptic Calabi-Yau three-folds are investigated well. It is known that a large class of $d=6$, $N=1$ superstring theories can be represented by F-theory compactification and they are conjectured to be dual to other string theories. As for compactifications on Calabi-Yau four-folds, the existence of non-perturbative superpotential of the four dimensional theory is determined by geometry [8]. In this way, F-theory compactification on Calabi-Yau 3,4-folds seems to be succeeded in itself. However, evidences such as stringy cosmic string solutions [9] corresponding to these compactifications are still lacking. In order to justify the F-theory compactification, it seems that there must be such solutions as in the case of eight dimensions, $F/K3$, since the compactification is originally defined as type IIB

compactification with certain properties.

Furthermore, if there exist a stringy cosmic string solution representing an F-theory compactification on a Calabi-Yau manifold, it gives explicit metric on the base manifold B and in a certain limit it is important in identifying the base manifold with, e.g., an orbifold.

In this article, we search for stringy cosmic string solutions corresponding to various F-theory compactifications by extending the method in ref.[9]. We obtain several solutions which are considered to represent F-theory on elliptic Calabi-Yau manifold with $B = \mathbf{CP}^1 \times \mathbf{CP}^1$ and with the Hirzebruch surface \mathbf{F}_n , and then we analyze their constant coupling limit. We also consider the compactification on Calabi-Yau three-fold with hodge numbers $h^{1,1}=h^{1,2}=19$ described in ref.[2].

This paper is organized as follows: First, we explain general properties of F-theory compactifications in section 2 and stringy cosmic string solutions corresponding to F-theory compactification on $K3$ in section 3. In section 4, extending the argument of section 3, we obtain stringy cosmic string solutions corresponding to F-theory compactifications on several elliptic Calabi-Yau 3,4-folds and consider their constant coupling limit. In section 5, we give conclusions briefly.

2 Compactifications of F-theory

F-theory is considered as a twelve dimensional theory underlying the conjectured $SL(2, \mathbf{Z})$ duality symmetry of type IIB theory in ten dimensions. Ten dimensional type IIB theory contains $g_{\mu\nu}$, $B_{\mu\nu}^A$, ϕ (NS-NS fields) and $B_{\mu\nu}^P$, $\tilde{\phi}$, $A_{\mu\nu\rho\sigma}^+$ (R-R fields) as bosonic fields in the low energy effective field theory. The field $\tau = \tilde{\phi} + i \exp(-\phi)$ transforms as the complex modulus of a torus under an $SL(2, \mathbf{Z})$ duality transformation.

Compactification of F-theory to lower dimensions is formulated as that of type IIB theory on a manifold B with τ varies according to a position in B [2]. More precisely, compactification of F-theory on a manifold X which admits elliptic fibration over a base B is type IIB compactification on B with $\tau = \tau(z^i)$ where $\{z^i\} \in B$.

In the case of elliptic $K3$, base manifold is \mathbf{CP}^1 and the elliptic fiber is given on each $z \in \mathbf{CP}^1$ in the Weierstrass form :

$$y^2 = x^3 + f_8(z)x + g_{12}(z) \quad (1)$$

where z is affine coordinate of \mathbf{CP}^1 , and f_8 and g_{12} are polynomials of degree ≤ 8 and ≤ 12 in z respectively. The elliptic fiber degenerates when the discriminant

$$\Delta = 4f^3 + 27g^2 \quad (2)$$

vanishes, which happens generically 24 points on \mathbf{CP}^1 . In the point of view of type IIB theory, this phenomenon represents that there are 24 D 7-branes located on \mathbf{CP}^1 and each of them carries a magnetic charge for $\tilde{\phi}$. The modulus τ of the fiber torus is given as

$$J(\tau(z)) = \frac{4(24f)^3}{4f^3 + 27g^2}. \quad (3)$$

Here $J(\tau)$ is the modular invariant function of τ defined as

$$J(\tau) = \frac{(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8)^3}{\eta(\tau)^{24}}. \quad (4)$$

Geometry of elliptic $K3$ can teach us the spectrum of the resulting eight dimensional theory, and the theory is conjectured to be dual to heterotic string theory compactified on T^2 [2, 4].

Similarly, we can define compactifications of F-theory on a Calabi-Yau three-fold X which admit elliptic fibration over a complex two dimensional base B [5]. If we choose B suitably, the resulting six dimensional theories have $N=1$ supersymmetry. The number of tensor multiplets T , vector multiplets V and hyper multiplets H are related to the geometry of X and B as

$$T = h^{1,1}(B) - 1 \quad (5)$$

$$r(V) = h^{1,1}(X) - h^{1,1}(B) - 1 \quad (6)$$

$$H = h^{2,1} + 1 \quad (7)$$

where $r(V)$ is the rank of the vector multiplet and V is determined by the singularity type of the fiber [5, 6].

For example, we can choose X as an elliptic fibration over the Hirzebruch surface \mathbf{F}_n , which is a \mathbf{CP}^1 bundle over \mathbf{CP}^1 characterized by an integer n . The surface \mathbf{F}_n is described as the quotient

$$(x, y, u, v) \sim (\lambda x, \lambda y, \mu u, \lambda^n \mu v) \quad (8)$$

with $\lambda, \mu \in \mathbf{C}^*$ where (x, y) and (u, v) are considered as homogeneous coordinates of base \mathbf{CP}^1 and fiber \mathbf{CP}^1 respectively. The elliptic fiber on \mathbf{F}_n is represented as

$$y^2 = x^3 + \sum_{k=-4}^4 f_{8-nk}(z') z^{4-k} x + \sum_{k=-6}^6 g_{12-nk}(z') z^{6-k} \quad (9)$$

where z is affine coordinate on the fiber \mathbf{CP}^1 and z' is that of the base \mathbf{CP}^1 . Polynomials f_{8-nk} and g_{12-nk} are of degrees $\leq 8-nk$ and $\leq 12-nk$ respectively and they are identical

to zero when the coefficients are negative. It can be seen from eq.(9) that the elliptic fiber degenerates on a codimension 1 surface Σ in B . There are a number (≤ 24) of connected components of Σ and each of them is identified with a part of D7-brane worldvolume. These theories have one tensor multiplet since $h^{1,1}(\mathbf{F}_n) = 2$. In the case of $2 \leq n \leq 12$, it is conjectured that the theory is conjectured to be dual to compactification of $E_8 \times E_8$ heterotic theory on $K3$ with instanton numbers $(12+n, 12-n)$ for each E_8 's after higgsing as much as possible [5]. In the case of $n = 0$, i.e., $B = \mathbf{CP}^1 \times \mathbf{CP}^1$, eq.(9) becomes

$$y^2 = x^3 + f(z, z')x + g(z, z') \quad (10)$$

where f (or g) is of degree ≤ 8 (or ≤ 12) in each z and z' and this manifold represents the same Calabi-Yau manifold as in the $n=2$ case.

Other than \mathbf{F}_n , we have a large choice of B in order to obtain six dimensional theories with $N=1$ supersymmetry [5].

Among others we describe one example for future convenience. That is an elliptically fibered Calabi-Yau manifold over \mathbf{CP}^2 blown up at nine points having hodge numbers $h^{1,1} = h^{1,2} = 19$. This manifold is also represented as [2, 5] two elliptic fibrations over \mathbf{CP}^1 as

$$y_i^2 = x_i^3 + f_i(z)x_i + g_i(z) \quad (i = 1, 2) \quad (11)$$

where $i=1, 2$ represent two fiber tori on \mathbf{CP}^1 and f_i (or g_i) is a polynomial of degree ≤ 4 (or ≤ 6).

3 Type IIB stringy cosmic string solutions in eight dimensions

One of the justifications of F-theory compactifications on $K3$ is that there exist stringy cosmic strings [9, 10] of type IIB low energy effective theory that reflect main properties of elliptic $K3$.

The low energy effective action of type IIB theory in ten dimensions with $B^A = B^P = A^+ \equiv 0$ is

$$S_{10} = -\frac{1}{2} \int d^{10}x \sqrt{g_{(10)}} \left(R_{(10)} - \frac{1}{2} \frac{\partial_\mu \tau g^{\mu\nu} \partial_\nu \bar{\tau}}{\tau_2^2} \right) \quad (12)$$

where $\tau = \tilde{\phi} + i \exp(-\phi) (\equiv \tau_1 + i\tau_2)$. This action is invariant under an $SL(2, \mathbf{Z})$ modular transformation:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}). \quad (13)$$

In order to obtain solutions of this action corresponding to compactifications of F-theory on elliptic $K3$ over the base \mathbf{CP}^1 , we set $\tau = \tau(z)$ where $z = x_8 + ix_9$ is affine coordinate on \mathbf{CP}^1 . Correspondingly, we take an ansatz for ten dimensional metric as

$$ds^2 = e^{\phi(z, \bar{z})} dz d\bar{z} + dx_7^2 + \cdots + dx_1^2 - dx_0^2. \quad (14)$$

Then the equations of motion of the action (12) are [9]

$$\partial \bar{\partial} \tau = \frac{2 \partial \tau \bar{\partial} \bar{\tau}}{\bar{\tau} - \tau}, \quad (15)$$

$$\partial \bar{\partial} \phi = \frac{\partial \tau \bar{\partial} \bar{\tau}}{(\tau - \bar{\tau})^2} (= \partial \bar{\partial} \log \tau_2). \quad (16)$$

The first equation eq.(15) is automatically satisfied since we set $\tau = \tau(z)$. The second equation is solved as

$$\phi = \log \tau_2 + F(z) + \bar{F}(\bar{z}) \quad (17)$$

where $F(z)$ is an arbitrary function of z . In order that eq.(14) makes sense as a metric, the field $\phi = \phi(\tau(z), z)$ must be invariant under the modular transformation of τ and e^ϕ cannot vanish on anywhere in \mathbf{CP}^1 since we want a solution such that τ is determined only up to an $SL(2, \mathbf{Z})$ transformation. If $\tau(z)$ is given by eq.(3) with²

$$\Delta = 4f^3 + 27g^2 \equiv C \prod_{i=1}^{24} (z - z_i), \quad (18)$$

these conditions imply

$$\exp(\phi(z, \bar{z})) = \tau_2 \eta^2 \bar{\eta}^2 \prod_{i=1}^{24} (z - z_i)^{-\frac{1}{12}} \prod_{i=1}^{24} (\bar{z} - \bar{z}_i)^{-\frac{1}{12}} \tilde{F}(z) \bar{\tilde{F}}(\bar{z}) \quad (19)$$

where $\tilde{F}(z)$ is regular and non-vanishing function of z and we take it to be a constant. The function η is Dedekind η -function :

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (q = \exp(2\pi i \tau)) \quad (20)$$

which is needed in eq.(19) to compensate modular transformation of τ_2 . This metric can be extended smoothly to infinity $z \rightarrow \infty$ in \mathbf{CP}^1 since as $z \rightarrow \infty$, $\tau \rightarrow \text{const.}$ and thus $e^\phi \rightarrow (z\bar{z})^{-2}$.

² When $\text{ord}(\Delta) < 24$, we can also obtain solutions corresponding to eq.(19). In this case however, we have to use $w = 1/z$ patch to indicate the metric around $z \rightarrow \infty$ at which the metric becomes singular. An example of this situation is given in the case of (22), as (23) and (24).

Note that

$$\tau(z) \sim \frac{1}{2\pi i} \log(z - z_i) \quad (z \rightarrow z_i) \quad (21)$$

and if we go around $z = z_i$, we have $\tau \rightarrow \tau + 1$. Thus we may say that at $z = z_i$ there is a D7-brane since it couples to magnetic dual for $\tilde{\phi}(=\tau_1)$ field.

The solution (19) represents important property of F-theory compactification such that at 24 points on the base τ_2 goes to infinity and the metric diverges. Thus the existence of such solutions gives us an evidence for F-theory. Or it can be said that the solutions support the notion of F-theory.

Note that stringy cosmic string solutions can teach us where D7-branes are located. However, they cannot predict symmetry enhancement. For example, in the F-theory point of view, the elliptic $K3$ with $E_8 \times E_8$ symmetry is represented as [5]

$$y^2 = x^3 + \alpha z^4 x + z^5 + \beta z^6 + z^7 \quad (22)$$

where α and β are constants. In this case, singular fibers of type E_8 appear at $z = 0$ and $z = \infty$ corresponding to the symmetry $E_8 \times E_8$. Besides, stringy cosmic string solutions representing eq.(22) in the z coordinate system is

$$\exp(\phi(z, \bar{z})) = a\tau_2\eta^2\bar{\eta}^2 \left| z^{-\frac{10}{12}} \prod_{i=1}^4 (z - z_i)^{-\frac{1}{12}} \right|^2. \quad (23)$$

Around $z = \infty$, taking the coordinates $w = 1/z$, we have

$$\exp(\phi) = \tilde{a}\tau_2\eta^2\bar{\eta}^2 \left| w^{-\frac{10}{12}} \prod_{i=1}^4 (w - \frac{1}{z_i})^{-\frac{1}{12}} \right|^2. \quad (24)$$

Here $z_i = z_i(\alpha, \beta)$ is defined as

$$\Delta = z^{10} (4\alpha^3 z^2 + 27(1 + \beta z + z^2)^2) \quad (25)$$

$$\equiv 27 z^{10} \prod_{i=1}^4 (z - z_i). \quad (26)$$

The two representations (23) and (24) are transformed to each other by usual coordinate transformation $z = 1/w$ if constants a and \tilde{a} are suitably chosen. The above solutions tell us that there are respectively ten D7 branes at $z = 0$ and $z = \infty$ and they give no explanation of enhanced symmetries.

4 Stringy cosmic string solutions and F-theory compactifications on elliptic Calabi-Yau manifolds

The purpose of this section is to solve the type IIB action (12) and find solutions representing F-theory compactified on Calabi-Yau three- or four-folds. Now, consider the

compactification of F-theory on elliptic Calabi-Yau three-folds over two dimensional base B . In the point of view of type IIB theory, this situation is represented by solving the equations of motion of the action (12) under the assumption

$$\tau = \tau(z, z') \quad (27)$$

where $z = x^8 + ix^9$ and $z' = x^6 + ix^7$. The metric is taken to be

$$ds^2 = ds_B^2 + dx_5^2 + \cdots dx_1^2 - dt^2. \quad (28)$$

We assume that the metric ds_B^2 on B is Hermitian :

$$ds_B^2 = g_{z\bar{z}}dzd\bar{z} + g_{z'\bar{z}'}dz'd\bar{z}' + g_{z\bar{z}'}dzd\bar{z}' + g_{z'\bar{z}}dz'd\bar{z} \quad (29)$$

where

$$\overline{g_{\alpha\beta}} = g_{\beta\bar{\alpha}}.$$

The equation of motion of the action (12) with respect to τ under the above assumption of the metric and the field τ is

$$\partial\tau\bar{\partial}g_{z'\bar{z}'} + \partial'\tau\bar{\partial}'g_{z\bar{z}} - \partial\tau\bar{\partial}'g_{z'\bar{z}} - \partial'\tau\bar{\partial}g_{z\bar{z}'} = 0 \quad (30)$$

where we use the notation

$$\partial \equiv \frac{\partial}{\partial z}, \quad \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}, \quad \partial' \equiv \frac{\partial}{\partial z'}, \quad \bar{\partial}' \equiv \frac{\partial}{\partial \bar{z}'} . \quad (31)$$

In contrast to the eight dimensional case, this equation gives complicated restriction on $\tau = \tau(z, z')$ and on ds_B^2 .

In order to simplify the problem, we take the ansatz $g_{z'\bar{z}} = 0$ which corresponds to taking a diagonal metric. We write

$$g_{z\bar{z}} = e^{\phi(z, \bar{z}, z', \bar{z}')} , \quad (32)$$

$$g_{z'\bar{z}'} = e^{\psi(z, \bar{z}, z', \bar{z}')} . \quad (33)$$

To meet the eq.(30) independent of τ , we further assume

$$\phi = \phi(z, \bar{z}), \quad \psi = \psi(z', \bar{z}'). \quad (34)$$

Next, we have to solve Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} \quad (35)$$

with

$$T_{\mu\nu} = \frac{1}{2} \frac{\partial_{(\mu} \tau \partial_{\nu)} \bar{\tau}}{\tau_2^2} - \frac{1}{4} \frac{\partial_\rho \tau g^{\rho\sigma} \partial_\sigma \bar{\tau}}{\tau_2^2} g_{\mu\nu}. \quad (36)$$

Under the assumptions we have made, the Einstein equations reduce to the following three equations :

$$0 = \frac{\partial\tau\bar{\partial}'\bar{\tau} \pm \partial'\tau\bar{\partial}\bar{\tau}}{\tau_2^2}, \quad (37)$$

$$\partial\bar{\partial}\phi = \frac{\partial\tau\bar{\partial}\bar{\tau}}{\tau_2^2}, \quad (38)$$

$$\partial'\bar{\partial}'\phi = \frac{\partial'\tau\bar{\partial}'\bar{\tau}}{\tau_2^2}. \quad (39)$$

The first equation leads

$$\partial\tau = 0 \quad \text{or} \quad \partial'\tau = 0, \quad (40)$$

and we take $\partial'\tau = 0$, i.e., $\tau = \tau(z)$. Then, eqs.(38) and (39) can be solved as

$$\phi = \log \tau_2 + F(z) + \bar{F}(\bar{z}), \quad (41)$$

$$\psi = F'(z) + \bar{F}'(\bar{z}). \quad (42)$$

Using the similar arguments as in the case of eight dimensions,

$$\exp(\phi(z, \bar{z})) = a\tau_2\eta^2\bar{\eta}^2 \left| \prod (z - z_i)^{-\frac{1}{12}} \right|^2 \quad (43)$$

$$\exp(\psi(z', \bar{z}')) = |F'(z')|^2 \quad (44)$$

where z_i represents zero point of $\Delta(z, z')$, i.e., $\Delta(z, z') = C(z') \prod (z - z_i)$ and $F'(z')$ is a non-vanishing function.

Now let us consider what happens when we restrict $\tau = \tau(z)$ in the F-theory point of view. As is described in section 2, if the base $B = \mathbf{F}_n$, polynomials

$$f(z, z') = \sum_{k=-4}^4 f_{8-nk}(z') z^{4-k}$$

and

$$g(z, z') = \sum_{k=-6}^6 g_{12-nk}(z') z^{6-k}$$

determine the structure of elliptic fiber τ as in eq.(3). There are the following three cases satisfying $\tau = \tau(z)$:

$$(a) \quad \begin{cases} f(z, z') = f_0(z)h(z')^2 \\ g(z, z') = g_0(z)h(z')^3 \end{cases} \quad (45)$$

$$(b) \quad f(z, z') = 0 \quad (46)$$

$$(c) \quad g(z, z') = 0. \quad (47)$$

The last two cases (b) and (c) correspond to the constant coupling solutions $\tau = \text{const.}$, and we discuss them later on. Here we consider the case (a). In this case the order of $f_0(z)$, $g_0(z)$ and $h(z')$ depend on the base B we choose. The discriminant is

$$\Delta = h(z')^6 (4f_0(z)^3 + 27g_0(z)^2). \quad (48)$$

The twelve dimensional metric is naturally taken to be the following modular invariant form [11]:

$$ds_{12} = h_{pq} dy^p dy^q + ds_B + ds_{6\mathcal{M}} \quad (49)$$

where $ds_{6\mathcal{M}}$ denotes the six dimensional Minkowski metric, y^p the coordinates of the fiber torus and

$$h_{pq} = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}. \quad (50)$$

Note however, that in order to obtain the theory with $N = 1$ supersymmetry in six dimensions, we should modify this metric such as internal metric on the elliptic Calabi-Yau manifold to be a Ricci-flat Kähler metric [9].

In the following, we consider the case $B = \mathbf{CP}^1 \times \mathbf{CP}^1 (= \mathbf{F}_0)$ and \mathbf{F}_n , respectively.

4.1 $B = \mathbf{CP}^1 \times \mathbf{CP}^1$

Now we consider the case of $B = \mathbf{CP}^1 \times \mathbf{CP}^1$. The coordinates z and z' are naturally taken to be those of two \mathbf{CP}^1 's. From eq.(9), the polynomial $f(z, z')$ (or $g(z, z')$) is generically of order 8 (or 12) in each of z and z' . If it is the case of (a), f_0 and g_0 are of order 8 and 12 in z respectively, and h is of order 4 in z' . Thus the discriminant and $\tau = \tau(z)$ are given explicitly as

$$\Delta = \{4f_0(z)^3 + 27g_0(z)^2\} h(z')^6 \quad (51)$$

$$\equiv C \prod_{i=1}^{24} (z - z_i) \prod_{i=1}^4 (z' - z'_i)^6 \quad (52)$$

and

$$J(\tau(z)) = \frac{4(24f_0(z))^3}{4f_0(z)^3 + 27g_0(z)^2}. \quad (53)$$

We see that the elliptic fiber degenerates on codimension 1 locus $\Delta = 0$ on B , i.e., on $z = z_i$ ($i = 1, \dots, 24$) and on $z' = z'_i$ ($i = 1, \dots, 4$). On the other hand, τ diverges on $z = z_i$ ($i = 1, \dots, 24$) which correspond to parts of D7-brane worldvolume.

Specializing (43) and (44) to this case, we naturally obtain

$$e^\phi = a\tau_2\eta^2\bar{\eta}^2 \prod_{i=1}^{24} (z - z_i)^{-\frac{1}{12}} \prod_{i=1}^{24} (\bar{z} - \bar{z}_i)^{-\frac{1}{12}}, \quad (54)$$

$$e^\psi = a' \prod_{i=1}^4 (z' - z'_i)^{-\frac{1}{2}} \prod_{i=1}^4 (\bar{z}' - \bar{z}'_i)^{-\frac{1}{2}}. \quad (55)$$

By the similar argument as in the case of eight dimensions, we see that both $e^\phi dz d\bar{z}$ and $e^\psi dz' d\bar{z}'$ represent the metric on \mathbf{CP}^1 globally since $e^\phi \rightarrow (z\bar{z})^{-2}$ as $z \rightarrow \infty$ and $e^\psi \rightarrow (z'\bar{z}')^{-2}$ as $z' \rightarrow \infty$. Therefore we conclude that we obtain the cosmic string solutions representing F-theory compactified on elliptic Calabi-Yau manifold over the base $B = \mathbf{CP}^1 \times \mathbf{CP}^1 \ni \{z, z'\}$ with $\tau = \tau(z)$. It can also be said that we have given explicit examples of compact D-manifolds [12].

Note that the above argument can easily be extended to the case of an elliptic Calabi-Yau four-fold over the base $B = \mathbf{CP}^1 \times \mathbf{CP}^1 \times \mathbf{CP}^1 \ni \{z, z', z''\}$ with $\tau = \tau(z)$.

4.2 $B = \mathbf{F}_n$

Various properties of F-theory compactified on elliptic Calabi-Yau manifolds with $B = \mathbf{F}_n$ such that spectrum or enhanced gauge symmetry have been closely investigated [5, 6]. We thus want to construct stringy cosmic string solutions reflecting these properties. We describe the results by taking an example $B = \mathbf{F}_4$. Other cases can be treated similarly.

We know from the structure of singular fiber on $B = \mathbf{F}_4$ that the theory has generically symmetry enhancement of $\text{SO}(8)$. The elliptic modulus $\tau(z, z')$ is given by eq.(9) with $n = 4$:

$$J(\tau(z, z')) = \frac{4(24f(z, z'))^3}{4f(z, z')^3 + 27g(z, z')^2} \quad (56)$$

$$f(z, z') = z^2 \sum_{k=0}^6 f_{4k}(z') z^k \quad (57)$$

$$g(z, z') = z^3 \sum_{k=0}^9 g_{4k}(z') z^k \quad (58)$$

where f_{4k} (or g_{4k}) is as usual a polynomial of order $\leq 4k$ in z' . Thus generically we may write

$$\Delta = Cz^6 \prod_{i=1}^{18} \{z + h_i(z')\}. \quad (59)$$

Here h_i is of order ≤ 4 . We restrict τ to depend only on one coordinate $y = y(z, z')$ in order that we can solve equations of motion of the action (12) easily. Choosing an order ≤ 4 function $\xi(z')$, we take $y = z + \xi(z')$ and $\tau = \tau(z + \xi(z'))$. This restriction corresponds to taking

$$f = \tilde{f}_6(y)z^2, \quad g = \tilde{g}_9(y)z^3. \quad (60)$$

By using a coordinate system $(y, y') = (z + \xi(z'), z')$, we assume the metric on B as

$$ds_B^2 = e^{\phi(y, \bar{y}, y', \bar{y}')} dy d\bar{y} + e^{\psi(y, \bar{y}, y', \bar{y}')} dy' d\bar{y}'. \quad (61)$$

The fields ϕ and ψ can be determined as the same way as in the case $B = \mathbf{CP}^1 \times \mathbf{CP}^1$:

$$\exp(\phi(y, \bar{y})) = a\tau_2\eta^2\bar{\eta}^2 \left| \prod_{i=1}^{18} (y - y_i)^{-\frac{1}{12}} \right|^2 \quad (62)$$

$$\exp(\psi(y', \bar{y}')) = |G(y')|^2 \quad (63)$$

where

$$\Delta = Cz^6 \prod_{i=1}^{18} (y - y_i). \quad (64)$$

Coordinate transformation back to (z, z') leads

$$ds_B^2 = e^\phi dz d\bar{z} + e^\phi \left(\frac{\partial \xi}{\partial z'} dz' d\bar{z} + c.c. \right) + \left(e^\phi \left| \frac{\partial \xi}{\partial z'} \right|^2 + e^\psi \right) dz' d\bar{z}'. \quad (65)$$

Furthermore, we have to take into account the effect of zero locus $z = 0$ of Δ corresponding to the symmetry $SO(8)$ (D_4 singularity) on $z = 0$. The result is

$$ds_B^2 = e^\phi |z|^{-1} dz d\bar{z} + e^\phi \left(\frac{\partial \xi}{\partial z'} \bar{z}^{-\frac{1}{2}} dz' d\bar{z} + c.c. \right) + \left(e^\phi \left| \frac{\partial \xi}{\partial z'} \right|^2 + e^\psi \right) dz' d\bar{z}'. \quad (66)$$

We can check that this metric certainly satisfy Einstein equations. This solution represents that τ_2 diverges on $y = y_i$ ($y_i = 1, \dots, 18$) and the metric diverges on $\Delta = 0$. These are the properties expected from F-theory point of view. If we fix a point z' on the base of \mathbf{F}_4 and take the limit $z \rightarrow \infty$, the metric becomes

$$e^\phi |z|^{-1} dz d\bar{z} \rightarrow (z\bar{z})^{-2} dz d\bar{z}.$$

Thus this solution seems to represent that the fiber of \mathbf{F}_4 is \mathbf{CP}^1 globally. However, note that contrary to the $B = \mathbf{CP}^1 \times \mathbf{CP}^1$ case, this solution is only applicable in the region $|z'| < \infty$. We do not know the way of extending this solution to cover all regions of \mathbf{F}_4 .

For other cases $B = \mathbf{F}_n$ we can obtain the similar solution if τ is restricted as $\tau = \tau(z + \xi_n(z'))$ where $\xi_n(z')$ is a polynomial of order $\leq n$.

4.3 Constant coupling solutions

It is known that a constant coupling limit $\tau(z) \rightarrow \text{const.}$ (up to an $SL(2, \mathbf{Z})$ transformation) of F-theory compactified on $K3$ is corresponding to orbifold limit of $K3 \rightarrow T^4/\mathbf{Z}_n$

with $n = 2, 3, 4$ or 6 [4, 13]. Similar considerations in the case of elliptic Calabi-Yau three-fold over $B = \mathbf{CP}^1 \times \mathbf{CP}^1$ have been done in ref.[15, 16, 17]. More generally, the relation between F-theory compactified on Calabi-Yau n -folds and type IIB orientifold theories have been investigated [18]. Here we consider F-theory compactifications on Calabi-Yau three-folds with τ remains constant in the point of view of stringy cosmic string solutions.

First, we consider the case $B = \mathbf{CP}^1 \times \mathbf{CP}^1$. In this case, since we already have solutions with $\tau = \tau(z)$ in section 4.1, we can obtain constant coupling solutions by taking a certain limit of them. There are the following three cases of obtaining constant coupling solutions:

$$(a) \quad f_0(z) \rightarrow \phi(z)^2, \quad g_0(z) \rightarrow \alpha \phi(z)^3 \quad (67)$$

$$(b) \quad g_0(z) \rightarrow 0 \quad (68)$$

$$(c) \quad f_0(z) \rightarrow 0. \quad (69)$$

They are respectively corresponding to the cases (a), (b) and (c) described in the first part of this section. Note that in the first case the value of τ is determined by the constant α whereas $\tau = i$ in the case (b) and $\tau = e^{i\pi/3}$ in (c). In the first case, we see from (54) and (55) that the metric on the base B becomes

$$ds_B^2 = a \prod_{i=1}^4 (z - z_i)^{-\frac{1}{2}} \prod_{i=1}^4 (\bar{z} - \bar{z}_i)^{-\frac{1}{2}} dz d\bar{z} + a' \prod_{i=1}^4 (z' - z'_i)^{-\frac{1}{2}} \prod_{i=1}^4 (\bar{z}' - \bar{z}'_i)^{-\frac{1}{2}} dz' d\bar{z}'. \quad (70)$$

We see that the topology of B is considered as $T^2/\mathbf{Z}_2 \times T^2/\mathbf{Z}_2$. It is claimed in ref.[14, 15] that this limit is related to F-theory compactified on $T^6/(\mathbf{Z}_2 \times \mathbf{Z}_2)$. Similarly, in the remaining cases, we can specify the metric on B and they can be interpreted as orbifold limit of F-theory compactifications.

Besides, independent of above three cases, we can construct several constant coupling solutions. For example, the metric

$$ds_B^2 = a \prod_{i=1}^3 (z - z_i)^{-\frac{2}{3}} \prod_{i=1}^3 (\bar{z} - \bar{z}_i)^{-\frac{2}{3}} dz d\bar{z} + a' \prod_{i=1}^3 (z' - z'_i)^{-\frac{2}{3}} \prod_{i=1}^3 (\bar{z}' - \bar{z}'_i)^{-\frac{2}{3}} dz' d\bar{z}' \quad (71)$$

corresponds to a solution with $\tau = e^{i\pi/3}$ and it is considered to be related to the F-theory compactified on $T^6/(\mathbf{Z}_3 \times \mathbf{Z}_3)$.

In the case of $B = \mathbf{F}_4$, taking $f(z, z') \rightarrow \phi(z, z')^2$ and $g(z, z') \rightarrow \alpha \phi(z, z')^3$ in eq.(66), the metric becomes

$$ds_{(B)}^2 \rightarrow \left| \prod_{i=1}^4 (z + \xi(z') - y_i)^{-\frac{1}{2}} \right|^2 |z|^{-1} dz d\bar{z} + \left| \prod_{i=1}^4 (z + \xi(z') - y_i)^{-\frac{1}{2}} \right|^2 \left(\frac{\partial \xi}{\partial z'} \bar{z}^{-\frac{1}{2}} dz' d\bar{z} + c.c. \right)$$

$$+ \left(\left| \prod_{i=1}^4 (z + \xi(z') - y_i)^{-\frac{1}{2}} \frac{\partial \xi}{\partial z'} \right|^2 + |G(z')|^2 \right) dz' d\bar{z}'. \quad (72)$$

In this case we have no information of determining the form of $G(z')$ and we cannot conclude that this limit corresponds to the orbifold limit of F-theory compactification at least from the point of view of stringy cosmic string solutions.

4.4 $h^{1,1} = h^{2,1} = 19$ model

We look for stringy cosmic string solutions corresponding to the $h^{1,1} = h^{2,1} = 19$ Calabi-Yau manifold described in eq.(11). This manifold is represented as an elliptic fibration over a base B . However, as we described in section 2, it is also interpreted as two equivalent tori T and \tilde{T} are fibered over the base \mathbf{CP}^1 . One of the two tori T is identified with the elliptic fiber over B and the other \tilde{T} is considered to be embedded in the base B . These fiber tori are explicitly represented as

$$y^2 = x^3 + f(z)x + g(z), \quad (73)$$

$$\tilde{y}^2 = \tilde{x}^3 + \tilde{f}(z)\tilde{x} + \tilde{g}(z) \quad (74)$$

where (x, y) and (\tilde{x}, \tilde{y}) denote the tori T and \tilde{T} respectively. The order of polynomials $f(z)$ and $\tilde{f}(z)$ is ≤ 4 and that of $g(z)$ and $\tilde{g}(z)$ is ≤ 6 . The discriminant of (73) and (74) are respectively

$$\Delta = 4f^3 + 27g^2, \quad (75)$$

$$\tilde{\Delta} = 4\tilde{f}^3 + 27\tilde{g}^2. \quad (76)$$

The torus T (or \tilde{T}) degenerates on $\Delta = 0$ (or $\tilde{\Delta} = 0$).

Now we look for stringy cosmic string solutions corresponding to the above situation. We start with the type IIB effective action in ten dimensions as before :

$$S_{10} = -\frac{1}{2} \int d^{10}x \sqrt{g_{(10)}} \left(R_{(10)} - \frac{1}{2} \frac{\partial_\mu \tau g^{\mu\nu} \partial_\nu \bar{\tau}}{\tau_2^2} \right). \quad (77)$$

We further compactify this on torus \tilde{T} . If we assume that the volume of two tori T and \tilde{T} is both fixed to 1, the resulting eight dimensional action after dimensional reduction becomes

$$S_8 = -\frac{1}{2} \int d^8x \sqrt{g_{(8)}} \left(R_{(8)} - \frac{1}{2} \frac{\partial_\mu \tau g^{\mu\nu} \partial_\nu \bar{\tau}}{\tau_2^2} - \frac{1}{2} \frac{\partial_\mu \tilde{\tau} g^{\mu\nu} \partial_\nu \bar{\tilde{\tau}}}{\tilde{\tau}_2^2} \right). \quad (78)$$

where we use

$$h_{ij} = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix} \quad (79)$$

as a metric on torus \tilde{T} .

In order to obtain solutions of the action, we take the ansatz on the eight dimensional metric as

$$ds_{(8)}^2 = e^{\phi(z, \bar{z})} dz d\bar{z} + ds_{6\mathcal{M}}^2 \quad (80)$$

where the coordinate $z = x_6 + ix_7$ represents the base \mathbf{CP}^1 . The total twelve dimensional metric is considered as

$$ds_{12}^2 = h_{pq} dy^p dy^q + \tilde{h}_{pq} d\tilde{y}^p d\tilde{y}^q + ds_{(8)}^2 \quad (81)$$

where

$$h_{pq} = \frac{1}{\tau_2} \begin{pmatrix} |\tau|^2 & \tau_1 \\ \tau_1 & 1 \end{pmatrix}, \quad \tilde{h}_{pq} = \frac{1}{\tilde{\tau}_2} \begin{pmatrix} |\tilde{\tau}|^2 & \tilde{\tau}_1 \\ \tilde{\tau}_1 & 1 \end{pmatrix}. \quad (82)$$

The equations of motion of the action (78) with respect to τ and $\tilde{\tau}$ have similar form as eq.(15) and they are satisfied by taking $\tau = \tau(z)$ and $\tilde{\tau} = \tilde{\tau}(z)$. The only independent equation in the Einstein equations is

$$\partial\bar{\partial}\phi = \frac{\partial\tau\bar{\partial}\bar{\tau}}{(\tau - \bar{\tau})^2} + \frac{\partial\tilde{\tau}\bar{\partial}\bar{\tilde{\tau}}}{(\tilde{\tau} - \bar{\tilde{\tau}})^2} \quad (83)$$

$$= \partial\bar{\partial}(\log \tau_2 + \log \tilde{\tau}_2) \quad (84)$$

which is solved as

$$\phi = \log \tau_2 + \log \tilde{\tau}_2 + F(z) + \bar{F}(\bar{z}). \quad (85)$$

As in the previous cases, the field ϕ must be invariant under modular transformations of both τ and $\tilde{\tau}$, and e^ϕ must be nonzero everywhere on \mathbf{CP}^1 . Thus,

$$\exp(\phi(z, \bar{z})) = a\tau_2\eta^2\bar{\eta}^2\tilde{\tau}_2\bar{\tilde{\tau}}^2\eta^2\bar{\eta}^2 \left| \prod_{i=1}^{12} (z - z_i)^{-\frac{1}{12}} \right|^2 \left| \prod_{j=1}^{12} (z - y_j)^{-\frac{1}{12}} \right|^2. \quad (86)$$

Here $\eta \equiv \eta(\tau)$ and $\tilde{\eta} \equiv \eta(\tilde{\tau})$ and z_i and y_i are given by

$$\Delta(z) \equiv C \prod_{i=1}^{12} (z - z_i), \quad (87)$$

$$\tilde{\Delta}(z) \equiv \tilde{C} \prod_{j=1}^{12} (z - y_j). \quad (88)$$

Since generically the order of Δ or $\tilde{\Delta}$ is 12, at infinity we have

$$e^{\phi(z, \bar{z})} dz d\bar{z} \rightarrow \frac{1}{(z\bar{z})^2} dz d\bar{z} \quad (|z| \rightarrow \infty). \quad (89)$$

Thus this metric indicates that the topology of the eight dimensional spacetime is certainly $\mathbf{CP}^1 \times M^6$. Each of $\Delta = 0$ locus $z = z_i$ corresponds to D7-brane worldvolume.

5 Conclusions

In this paper, we have solved equations of motion of low energy effective action of ten dimensional type IIB theory and have obtained stringy cosmic string solutions corresponding to compactifications of F-theory on elliptic Calabi-Yau 3,4-folds. In concrete, we have constructed solutions in the case of $B = (\mathbf{CP}^1)^{\otimes n}$, \mathbf{F}_n and a blown-up \mathbf{CP}^2 , and have investigated their constant coupling limits. For each of these manifolds, our solutions do not cover whole moduli space of the F-theory compactifications, which is caused by technical reasons. However, solutions corresponding to more general cases should exist.

Existence of such solutions is considered as one of the evidences of F-theory compactifications. In particular, the solutions explicitly show the codimension 1 surface in the base B representing the locations of D7-branes.

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